Pure & Appl. Chem., Vol.55, No.5, pp.845-852, 1983.
Printed in Great Britain.

0033-4545/83/050845-08$03.00
Pergamon Press Ltd.

FAILURE THEORIES FOR GLASSY POLYMERS:THE T-CRITERION FOR VARIOUS YIELD LOCI

P.S. Theocaris and N.P. Andrianopoulos

Department of Theoretical and Applied Mechanics, The National Technical University of Athens, 5, Heroes of Polytechnion Avenue, Zographou, Athens 624, Greece

Abstract - In the present paper a new fracture criterion, suitable for both brittle and ductile materials, is described. The main idea of this criterion is that, whichever is the critical quantity for crack initiation, it must be evaluated along the elastic-plastic boundary around the crack-tip. On the one hand, the introduction of the elastic-plastic boundary permits the direct evaluation of the size of the, so called, *"core region"*, and on the other hand the yield condition can be selected properly, to describe the brittle or ductile behaviour of the specific material. Experimental results from two glassy polymers (PMMA and PCBA) show good agreement with the theoretical predictions, especially in the case of the ductile material.

INTRODUCTION

Many attempts have, up to now, been made for the prediction of the direction to which a crack will propagate and the external load needed to initiate the fracture phenomenon. These attempts have resulted to three main fracture criteria, the, so-called σ_ϑ-criterion (Ref.1), S-criterion (Ref.2) and G-criterion (Refs.3 & 4).

All these criteria are applied in the same manner, the difference between them being in the selection of the characteristic mechanical quantity, which is considered as the critical one. Once the critical quantity is selected, its value is computed along a circle centered on the crack tip, by using the respective expressions of the singular stress field around it. The angle of crack extension coincides with the direction at which the critical quantity possesses an extremum. The value of this extreme, depending on the external load, is compared with a critical value (being a material constant). If the external load is high enough, the extreme value is equal to the critical one and the crack propagates to the expected direction.

However, the introduction of the expressions of the singular stress field for the evaluation of the critical quantity implies some limitations. On the one hand, these expressions are obtained under the assumption of a linear elastic material loaded statically or dynamically. In both types of load the linearity of the material is violated in areas in the close vicinity of the crack-tip. Even in the case of the brittlest material an amount of plasticity or non-linearity must exist. To avoid non-linearity a minimum radius r_0 of the circle around the crack-tip must be defined. This minimum circle was named as *"core region"* (Ref.5). On the other hand, the existence of core region raises questions on the nature of this region. Is this region circular? Are there any quantitative estimations of its magnitude?

It was, silently, supposed that in the case of brittle materials the small size of the non-linear area justifies an approximation by a circle. Although, this approximation is apparently coarse, it gave acceptable results in case of brittle materials. However, the only known quantitative approximation of the core region by a circle is that by the *initial curve* of the caustics (Refs.6 & 7).

But, ductile materials, like many of glassy polymers, can also fail by fracture. In this case the assumptions of linear elasticity cease to hold at much more longer distances compared with those of brittle materials. Then the approximation of the core region by a circle is unacceptably rough in the case of ductile materials.

Surprisingly enough, a fracture criterion similar to the existing ones and suitable for ductile materials did not exist. In a few recent papers (Refs.8,9 & 10) such an attempt was undertaken, which resulted in the introduction of the *T-criterion*.

In this paper the T-criterion will be presented and its flexibility will be shown.

THE BASIS OF THE T-CRITERION

Let consider the typical case of a thin infinite elastic plate containing a slant crack of length 2a and subjected to uniaxial tension σ_0 at infinity (Fig. 1). Generalized plane stress conditions are supposed. If the material is ductile, a part of the total energy T offered by the external forces, will be consumed by plastic deformations, the remaining being stored as elastic energy by elastic deformations of the material. Any increase of the externally offered energy will drive to an expansion of the plastic zone and an increase of the elastic

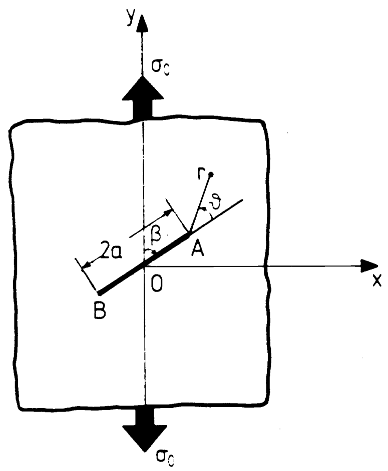

Fig. 1. A slant crack under tensile load at infinity.

energy stored. According to von Mises (Ref.11) yield-condition a module of the energy consumed for plastic deformations is the distortional part T_D of the total energy. Consequently, the dilatational part T_V of the total energy is available to cause fracture. T_V is a module of the normal stresses acting in the material. According to the most modern models of ductile fracture (Refs.12,13 & 14) normal stresses are responsible for this phenomenon by initiating void growth and coalescence, chain rearrangement and break through crazes in polymers and other fracture processes. Then, one has to define the maximum of T_V along the elastic-plastic boundary. The above remarks can be summarized as follows:

i) *A crack starts to propagate when the dilatational strain-energy density* T_V *at a point in the vicinity of its tip reaches a critical value* $T_{V,0}$.

ii) *As curve of evaluation of* T_V *around the crack-tip the elastic-plastic boundary is used, as it is obtained from the Mises yield-condition* $T_D = T_{D,0}$.

Algebraically, the above two hypotheses are described as:

$$T_V(r,\vartheta)\Big|_{\vartheta=\vartheta_0} \geq T_{V,0} \quad , \quad \frac{\partial T_V}{\partial \vartheta}\Big|_{\vartheta=\vartheta_0} = 0$$
$$T_D(r,\vartheta)\Big|_{r=r(\vartheta)} = T_{D,0} \quad , \quad \frac{\partial^2 T_V}{\partial \vartheta^2}\Big|_{\vartheta=\vartheta_0} < 0 \qquad (1)$$

The dilatational and distortional parts of strain-energy density are given respectively by:

$$T_V = \frac{1-2\nu}{6E}(\sigma_x+\sigma_y)^2 \qquad (2)$$

$$T_D = \frac{1+\nu}{3E}(\sigma_x^2+\sigma_y^2-\sigma_x\sigma_y+3\tau_{xy}^2) \qquad (3)$$

where σ_x, σ_y, τ_{xy} are the singular stresses around the crack-tip and (E,ν) the modulus of elasticity and Poisson's ratio of the material respectively.

Introducing into Eqs. (2) and (3) the expressions of σ_x, σ_y, τ_{xy}, as they are given in Ref.15, after some algebra, we obtain:

$$T_V = \frac{(1-2\nu)K_I^2}{12\pi Er}\{f_x(\vartheta) + f_y(\vartheta)\}^2 \qquad (4)$$

$$T_D = \frac{(1+\nu)K_I^2}{6\pi Er}\{f_x^2(\vartheta)+f_y^2(\vartheta)-f_x(\vartheta)f_y(\vartheta)+3f_{xy}^2(\vartheta)\} \tag{5}$$

where:

$$\left.\begin{aligned} f_x(\vartheta) &= \cos\vartheta/2-1/2\sin\vartheta\sin3\vartheta/2-\mu(2\sin\vartheta/2+1/2\sin\vartheta\cos3\vartheta/2 \\ f_y(\vartheta) &= \cos\vartheta/2+1/2\sin\vartheta\sin3\vartheta/2+1/2\mu\sin\vartheta\cos3\vartheta/2 \\ f_{xy}(\vartheta) &= 1/2\sin\vartheta\cos3\vartheta/2+\mu(\cos\vartheta/2-1/2\sin\vartheta\sin3\vartheta/2) \end{aligned}\right\} \tag{6}$$

In the above equations (r,ϑ) are the polar coordinates around the crack-tip, $\mu=K_{II}/K_I$, and (K_I, K_{II}) are the mode-I and -II stress intensity factors given by:

$$K_I = \sigma_0(\pi a)^{1/2}\sin^2\beta\ ,\quad K_{II} = \sigma_0(\pi a)^{1/2}\sin\beta\cos\beta \tag{7}$$

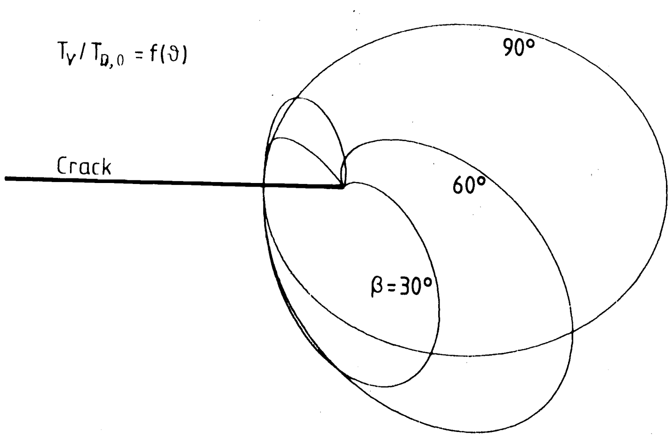

Fig. 2. Polar distribution of the reduced value $T_V/T_{D,0}$ of the dilatational strain energy density around the crack tip for $\beta=30^o$, 60^o, 90^o.

From Eq.(5) by putting $T_D=T_{D,0}$, we obtain for the radius of the elastic-plastic boundary:

$$r = \frac{(1+\nu)K_I^2}{6\pi ET_{D,0}}\{f_x^2(\vartheta)+f_y^2(\vartheta)-f_x(\vartheta)f_y(\vartheta)+3f_{xy}^2(\vartheta)\} \tag{8}$$

In Fig. 2 the angular distribution of the ratio $T_V/T_{D,0}$ is plotted for $\beta=30^o$, 60^o, 90^o. As it is seen this ratio possesses a maximum to the direction ϑ_0 of the expected crack propagation.

In the next Fig. 3 the shape of the elastic-plastic boundary is plotted, as it is given from Eq. (8).

In Fig.4 the quantities $T=T_V+T_D$, T_D and T_V, reduced to the constant quantity $T_{D,0}$, are plotted. Obviously, $T_D/T_{D,0}$ is a circle, because T, T_D, T_V are computed along the elastic-plastic boundary where $T_D=T_{D,0}$.

However, from a physical point of view, there is no unavoidable reason to use exclusively the Mises yield-condition for the description of the elastic-plastic boundary. Such statements, like the Mises condition, are axiomatically imposed and experimentally verified, but the same holds for similar statements as the Tresca or Mohr conditions. The core of T-criterion is that *the dilatational strain-energy density causes fracture outside the plastically deformed zone*. Hence, the second of Eqs (1) can be replaced by any physically equivalent yield condition.

From the classical Strength of Materials, there are two additional yield-conditions, similar to that of Mises. These are the Tresca and the Mohr yield-conditions (Ref.11). The first of them states that *"a material fails when the maximum shear stress reaches a critical value"* and the second that *"a suitable combination of normal and shear stresses, acting at a point,*

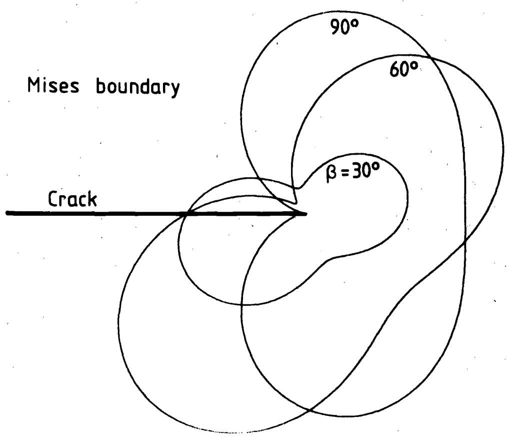

Fig. 3. The shape of the elastic-plastic boundary, according to Mises yield-condition, around the crack-tip for β=30°, 60°, 90°.

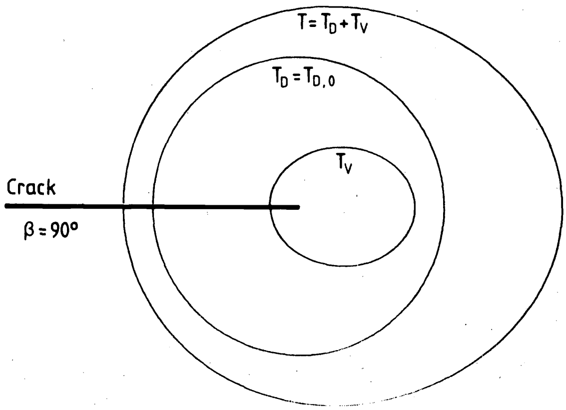

Fig. 4. Polar distribution of T_V, T_D, T around the crack-tip, evaluated along the Mises elastic-plastic boundary.

is necessary to produce yielding".

It is interesting to note that the Mises and Tresca conditions are generally referred to ductile failures and the Mohr condition to brittle ones.

The Tresca yield-condition is described in the case of generalized plane stress by the relation:

$$|\sigma_1 - \sigma_2| \geq C \tag{9}$$

where σ_1, σ_2 are the principal stresses at the point (r,ϑ) and C a material constant.

The respective Mohr yield-condition can be given in the form:

$$\sqrt{\sigma^2+\tau^2} \geq F(c_1, c_2,\ldots,c_n) \tag{10}$$

where σ and τ are the normal and shear stresses at a point (r,ϑ) of the material and $F(c_1, c_2,\ldots,c_n)$ an unknown function (the Mohr envelope), characterizing the material and depending on the relative magnitude of σ and τ, the internal friction of the material, its toughness in simple tension and simple shear and other factors reflecting the mode of structure of the material. This function is multivalued for each material and it is determined experimentally. The multivaluedness of this function gives rise to some difficulties in the general application of the Mohr condition but, on the other hand, permits specific modifications to fit to the behaviour of a specific material. However, as a first step, this function can be considered as a one-valued material constant.

Now, after some algebra, Eqs (9) and (10) give the following expressions for the polar radius of the elastic-plastic boundary:

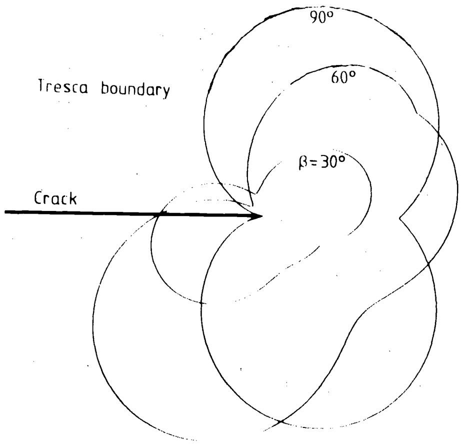

Fig. 5. The shape of the elastic-plastic boundary, according to Tresca yield-condition, around the crack-tip for β=30°, 60°, 90°.

For Tresca: $$r = \frac{8K_I^2}{\pi C^2}[(f_x-f_y)^2+4f_{xy}^2] \qquad (11)$$

For Mohr: $$r = \frac{2K_I^2}{\pi F^2}[(f_x+f_y)^2+4((f_x-f_y)^2+4f_{xy}^2)^{\frac{1}{2}}] \qquad (12)$$

Symbols f_x, f_y and f_{xy} represent the functions $f_x(\vartheta)$, $f_y(\vartheta)$ and $f_{xy}(\vartheta)$ given by Eq. (6).

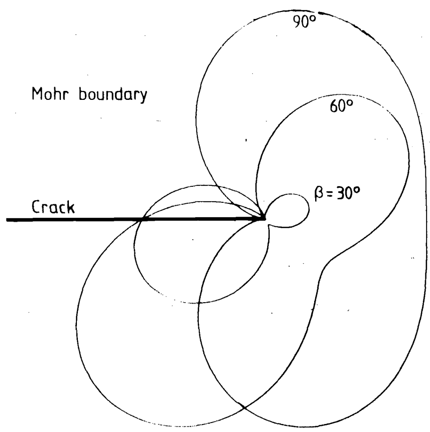

Fig. 6. The shape of the elastic-plastic boundary, according to Mohr yield-condition around the crack-tip for β=30°, 60°, 90°.

In Fig. 5 the elastic-plastic boundary according to the Tresca yield-condition is plotted for angles β=30°, 60°, 90° and in the next Fig. 6 the same boundary is traced according to the Mohr yield-condition. Comparison between Figs.3,5 and 6 indicates a considerable similarity between the results of the three yield-conditions, disregarding a scaling factor.

In Fig. 7 the predictions of the expected angle ϑ_0 of crack propagation are plotted according to the T-criterion for the three different descriptions of the elastic-plastic boundary, versus angle β of crack inclination. As it can be seen from this figure the predictions of the T-criterion, when either the Mises or the Tresca yield-conditions are used almost coincide, showing a maximum difference of ~2° for β→0°. On the contrary, the values of ϑ_0 in the case of the Mohr yield-condition are absolutely much smaller, showing a difference of ~20° from the previous predictions for β→0°. It is worth noting here that the latter predictions of ϑ_0 are similar to the predictions of S-criterion, which is a typical criterion for brittle fracture.

It is worth at this point making some remarks on the existing experimental data for the angle ϑ_0 of crack propagation. All the existing criteria and experimental evidence for both brittle (Refs.16 & 17) and ductile (Ref.9) materials more or less coincide for $\beta>45^o$.

Large theoretical deviations and experimental scattering are observed for small values of β ($\beta<45^o$). As a general trend, it can be said that brittle polymers (like PMMA) fracture under absolutely smaller angles, as compared to those of ductile ones (like PCBA).

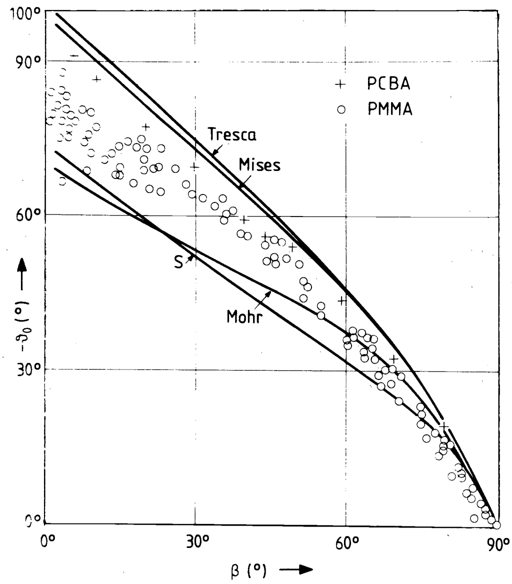

Fig. 7. Angle ϑ_0 of crack-initiation versus angle β, according to S- and T-criteria, for the three yield loci. Experimental points for PMMA were obtained from Refs.16 & 17 .

This trend is predicted in a natural way be the T-criterion, since, according to the reasoning of this criterion, the transition from a brittle to a ductile material and vice-versa, is easily described by a suitable selection of the yield-condition.

A second remark on the experimental data, is that the scattering in the values of ϑ_0 is much stronger in the case of brittle polymers. This phenomenon is very limited in ductile polymers, perhaps due to the considerable size of the plastic zone, which serves as a buffer of unstable situations, by smoothening extreme stress values.

A third remark is that for $\beta<10^o$, angle ϑ_0 can exceed -90^o. It is predicted by the T-criterion and, clearly measured in PCBA specimens (Ref.9). Values of ϑ_0 approaching -90^o were, also, measured in PMMA (Refs.16 & 17).

The second and most important quantity in the fracture phenomenon is the critical load. The predictions of T-criterion for this quantity are plotted in Fig. 8 versus the angle β of crack inclination. As it was expected, these predictions almost coincide, when the two ductile conditions (Tresca and Mises) are used for the description of the elastic-plastic boundary. Really, their maximum difference is less than 9% for $\beta=2^o$. Also, as the angle β of crack inclination varies from 90^o to 0^o, the relative increase of the critical load is ~53% in the case of the Mises yield-condition and ~67% in the case of the Tresca yield-condition.

These predictions are in strong disagreement with those of all the previous criteria. For example, the ratio $\sigma_0^\beta/\sigma_0^{90}$ equals to ~30_o in the case of S-criterion, instead of 1.53 or 1.67 in the case of the T-criterion when $\beta=2^o$.

However, experimental data from PCBA specimens for the critical fracture load, showed an excellent agreement with the predictions of the T-criterion, as it can be seen in the same figure.

The respective data from PMMA specimens show a stronger increase as angle β decreases, but not so high as the predictions of the previous criteria.

The unknown form of the Mohr envelope $F(c_1,c_2,...,c_n)$, introduced in Eq. (10) does not permit,

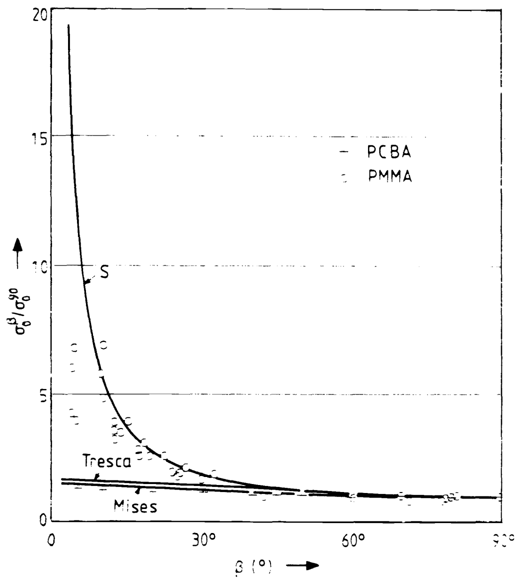

Fig. 8. Critical stress for crack-initiation $\sigma_0^\beta/\sigma_0^{90}$ versus angle β according to S- and T-criteria, for the Mises and Tresca yield loci. Experimental points for PMMA were obtained from Refs.16 & 17 .

for the present, the computation of the load predictions, when the Mohr yield-condition is used. The value of this function depends, among other parameters, on the relative balance between normal and shear stresses, i.e. it depends on angle β. This fact permits the computation of the elastic-plastic boundary, because, for a given β, F has a constant value and, thus, it acts as a scaling factor, not affecting the shape of the elastic-plastic boundary. On the contrary, the computation of $\sigma_0^\beta/\sigma_0^{90}$ asks for absolute values of the polar radius of the boundary, which depend directly on the value of $F(c_1,c_2,...,c_n)$.

In general, similar remarks, to those for the behaviour of angle ϑ_0, hold for the critical load. The experimental scattering of critical load is much more weak in ductile polymers, when compared to that of brittle ones and the increase of the critical load is much lower, as angle β decreases.

Finally, it seems that the T-criterion is a good approximation of the fracture phenomenon and wide enough to cover the behaviour of the materials from the brittle to ductile ones.

CONCLUSIONS

In the present paper a new fracture criterion was described. It is based on the idea that non-linearity or plasticity, always, exists even in the brittlest material. This physical assumption permits an insight to the complicated and dark nature of the core region around the crack-tip and drives to a direct evaluation of this region.

On the other hand, the T-criterion connects classical ideas, like the yield-conditions, with new concepts on fracture, like void growth and chain break, fact which indicates that old ideas are, still, fruitful.

The predictions of the T-criterion are sensitive to the specific type of failure and, thus, cover the behaviour of all the materials from the more ductile to the brittlest ones. Furthermore, the already existing criteria can be considered as limiting cases of the T-criterion. Moreover, the flexibility of adaptation to the specific behaviour of each material is a unique property of the T-criterion.

Experimental evidence obtained from a ductile polymer showed a clear preference in favour of the predictions of this criterion. Its predictions for brittle materials, being similar to those of existing criteria, are equally good or bad with the predictions of these criteria.

REFERENCES

1. Erdogan F. and Sih G.C., *J. of Basic Engng.*, 85D, pp. 519-527, (1963).
2. Sih, G.C., *Mechanics of Fracture*, 1, Noordhoff, Leyden, (1973).
3. Hussain, M.A., Pu S.L. and Underwood J., *ASTM, STP 560*, pp. 2-28, (1974).
4. Palaniswamy, K., Knauss E.G., *Int. J. Fract. Mech.*, pp. 114-117, (1972).
5. Sih, G.C., *Int. J. Fract. Mech.*, 10, pp. 305-321, (1974).
6. Theocaris, P.S., *Engng. Fract. Mech.*, 14, pp. 353-362, (1981).
7. Theocaris, P.S., *Mixed Mode Crack Propagation*, Sijthoff and Noordhoff, pp. 21-36, (1980).
8. Theocaris, P.S. and Andrianopoulos N.P., *Engng. Fract. Mech.*, 16, pp. 425-432, (1982).
9. Theocaris, P.S., Kardomateas G. and Andrianopoulos, N.P., *Engng. Fract. Mech.*, (1982) (to appear).
10. Theocaris, P.S. and Andrianopoulos N.P., *Int. J. Fract. Mech.*, (1982), (to appear).
11. Hill, R., *The Mathematical Theory of Plasticity*, Clarendon Press, (1964).
12. McClintock, F.A., *J. Appl. Mech.*, 35, pp. 363-371, (1968).
13. Rice, J.R. and Tracey D.M., *J. Mech. Phys. Solids*, 17, pp.201-217, (1969).
14. Gurson, A.L., *J. Engng. Mat. and Technology*, 99, pp. 2-15, (1977).
15. Paris, P.C. and Sih, G.C., *ASTM STP 381*, pp. 30-83, (1964).
16. Williams, J.G. and Ewing P.D., *Int. J. Fract. Mech.*, 8, pp. 441-446, (1972).
17. Sih, G.C. and Kipp M.E., *Int. J. Fract. Mech.*, 10, pp. 261-265, (1974).